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SHORTER COMMUNICATION

PLANAR RADIATING FLOW BY THE METHOD OF DISCRETE ORDINATES

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NOMENCLATURE

a_k, weighting coefficient;
a_s, isentropic speed of sound, $a_S \equiv \sqrt{(\gamma R T_\infty)}$;
a_T, isothermal speed of sound, $a_T \equiv \sqrt{(R T_\infty)}$;
Bo, Boltzmann number, $Bo_j \equiv \rho_{\infty j} u_{\infty j} R\gamma/(\gamma - 1)\sigma T^3_{\infty j}$ and $Bo \equiv \gamma R \rho_\infty a_S/(\gamma - 1)\,\sigma T^3_\infty$;
Bu, Bouguer number, $Bu = \alpha_\infty a_S/\omega$;
D_l, differential operator, $D_l \equiv (-\mu_l^2 \partial^2/\partial x^2 + \alpha^2_\infty)$;
I_k, radiation intensity at a particular direction;
I_0, averaged radiation intensity;
M, Mach number;
n, one half of total number of discrete directions; also the order of approximation;
P, thermodynamic pressure;
$P_{2n}(\mu)$, Legendre polynomial of $2n$ order;
q, radiation heat flux;
R, gas constant;
r_d, diffuse reflectivity of the wall;
r_s, specular reflectivity of the wall;
T, temperature of the gas;
T_w, wall temperature;
t, time;
u, velocity;
W_S, isentropic wave operator, $W_S \equiv \partial^2\phi/\partial t^2 - a_S^2\,\partial^2\phi/\partial x^2$;
W_T, isothermal wave operator, $W_T \equiv \partial^2\phi/\partial t^2 - a_T^2\,\partial^2\phi/\partial x^2$;
x, coordinate;
α, extinction coefficient, $\alpha \equiv \alpha_a + \alpha_s$;
α_a, absorption coefficient;
α_S, scattering coefficient;
β_j, dimensionless parameter, $\beta_j \equiv Bo_j(1 - M^2_{\infty j})/16(1 - \gamma M^2_{\infty j})$;
γ, ratio of specific heats;
ε, emissivity of the wall;
η_j, dimensionless coordinate, $\eta_j \equiv \alpha_{\infty j} x$;
μ, direction cosine of the angle between the direction of propagation of radiation and the positive x-axis;
μ_k, discrete direction cosine;
ξ, dimensionless coordinate, $\xi \equiv \omega x/a_s$;
ρ, gas density;
σ, Stefan–Boltzmann constant;
ϕ, perturbation velocity potential, $u' = \partial\phi/\partial x$, $p' = -\rho_\infty\,\partial\phi/\partial t$;
ω, radian frequency of oscillation;
ω_j, albedo of scattering, $\omega_j \equiv \alpha_{s\infty j}/\alpha_{\infty j}$.

Subscript
∞, conditions at infinity.

Superscript
$'$, disturbed quantities.

1. INTRODUCTION

THE ACCURACY of the differential approximation (1) in radiative gasdynamics has recently attracted considerable attention. In a recent paper by Cheng and Leonard [2], it is established that, for planar configuration, the error of the differential approximation is within 15 per cent for all flow and radiative quantities if external radiation is unimportant. For problem with external radiation, the error is also within 15 per cent for all variables except density which is rather inaccurate. For problems other than planar configuration,

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it is known that the error becomes more severe [3]. For this reason a number of papers trying to improve the accuracy of the differential approximation have recently appeared [4–6]. Although results based on these newly proposed methods do show improvements over that of the differential approximation for problems of simple geometry, the application of these methods to a multi-dimensional configuration is not straightforward.

A method which appears promising for treating both of the planar and non-planar multi-dimensional problems in radiative gasdynamics when a high degree of accuracy is required is that of the discrete-ordinate method. Although the application of the method has been limited previously to problems in radiation equilibrium [7], neutron transport [8], and rarefied gasdynamics [9] with planar or spherical symmetric configurations, the extension of the method to two-dimensional problems in rarefied gasdynamics has recently been discussed by Huang [10].

In this paper we shall discuss the method of discrete ordinates within the content of planar radiative gasdynamics. We shall obtain analytic solutions based on this method for the problems of linearized shock wave and the propagation of periodic disturbances where both of the exact and approximate solutions have been found [1, 2, 11, 12]. Numerical results evaluated up to the third approximations for both problems are compared with exact solutions.

2. METHOD OF DISCRETE ORDINATES

For planar configuration the method of discrete ordinates consists in (i) dividing the radiation field $(-1 < \mu < 1)$ into $2n$ streams in the discrete directions μ_k $(k = \pm 1, \pm 2, \ldots, \pm n)$ so that the radiation intensity in that particular direction is given by $I_k(x)$ where $I_k(x) \equiv I(x, \mu_k)$, and (ii) approximating integrals of the form

$$\int_{-1}^{1} f(x, \mu)\,d\mu = \sum_{k=\pm 1}^{\pm n} a_k f(x, \mu_k), \qquad (1)$$

by a sum with a_k's denoting the weighting coefficients which can be evaluated once for all if the division points μ_k's are specified (the values of a_k and μ_k for Gaussian quadrature is tabulated in [7]).

According to equation (1) the average radiation intensity and radiation heat flux can be approximated by

$$I_0(x) = 2\pi \int_{-1}^{1} I(x, \mu)\,d\mu \simeq 2\pi \sum_{p=\pm 1}^{\pm n} a_p I_p(x), \qquad (2)$$

$$q(x) = 2\pi \int_{-1}^{1} I(x, \mu)\,\mu\,d\mu \simeq 2\pi \sum_{p=\pm 1}^{\pm n} a_p \mu_p I_p(x), \qquad (3)$$

and the radiation transport equation (for an isotropic scattering grey gas in local thermodynamic equilibrium) is given by

$$\mu_k \frac{dI_k}{dx} + \alpha I_k = \frac{\alpha_s}{2} \sum_{p=\pm 1}^{\pm n} a_p I_p + \frac{\alpha_a \sigma T^4}{\pi}, \qquad (4)$$

where the temperature of the gas is not known a prior in radiative gasdynamics and must be determined simultaneously with other conservation equations.

There are two types of radiative boundary conditions which frequently occur in radiative gasdynamics.

(i) For a semi-infinite expanse of radiating gas occupying the positive x-direction with a reflecting wall at $x = 0$, the radiative boundary condition in discrete ordinates is [13]

$$I_k(0) = \varepsilon\sigma T_w^4/\pi + r_s I_{-k}(0) - 2r_d \sum_{p=1}^{n} I_{-p}(0)\mu_{-p} a_{-p}. \qquad (5)$$

$$(k = \pm 1, \pm 2, \ldots, \pm n).$$

(ii) For two radiating gases occupying adjacent spaces under different conditions with the assumptions that no radiation is reflected at the interface and the index of refraction is unity, the radiation intensity at the interface is continuous. Thus we have

$$I_k(0^+) = I_k(0^-), (k = \pm 1, \pm 2, \ldots, \pm n) \qquad (6)$$

where 0^- and 0^+ denote respectively the conditions immediately in front of and behind the interface.

It is evident that equations (3) and (4) with boundary condition (5) or (6) plus the conservation equations of mass, momentum and energy are a set of equations and boundary conditions in purely differential form.

3. NORMAL SHOCK WAVE

For the problem of a linearized shock wave located at $x = 0$, the governing equations in discrete ordinate form are

$$\beta_j \frac{dT'_j}{dx} + \alpha_{a\infty j} T'_j = \frac{\pi \alpha_{a\infty j}}{8\sigma T^3_{\infty j}} \sum_{p=\pm 1}^{\pm n} a_p I'_{jp} \qquad (7)$$

$$\mu_k \frac{dI'_{jk}}{dx} + \alpha_{\infty j} I'_{jk} = \frac{\alpha_{s\infty j}}{2} \sum_{p=\pm 1}^{\pm n} a_p I'_{jp} + 4\sigma\alpha_{a\infty j} T^3_{\infty j} T'_j/\pi,$$

$$(k = \pm 1, \pm 2, \ldots, \pm n) \qquad (8)$$

with boundary condition given by the linearized form of equation (6), that is

$$I'_{1k}(0^-) + \sigma T^4_{\infty 1}/\pi = I'_{2k}(0^+) + \sigma T^4_{\infty 2}/\pi,$$

$$(k = \pm 1, \pm 2, \ldots, \pm n) \qquad (9)$$

and that the disturbances must vanish at infinity.

The solution of the problem is then given by

$$\frac{I'_{jk}(\eta_n)}{4\sigma T^4_{\infty 2}/\pi} = \sum_{m=1}^{n} g_{jm,k} \exp(-c_{jm}\eta_j),$$

$$(k = \pm 1, \pm 2, \ldots, \pm n) \qquad (10)$$

and

$$\frac{T'_j(\eta_j)}{T_{\infty 2}} = \sum_{m=1}^{n} h_{jm} \exp(-c_{jm}\eta_j), \qquad (11)$$

where $\pm c_{jm}$ is determined from

$$\sum_{k=1}^{n} \frac{a_k}{1-\mu_k^2 c_{jm}^2} = \frac{1-\omega_j - \beta c_{jm}}{1-\omega_j-\omega_j\beta_j c_{jm}}, \tag{12}$$

with non-zero negative root for c_{1m} and non-zero positive root for c_{2m}. h_{jm} and $g_{jm,k}$ are given by

$$\sum_{m=1}^{n}\left\{\frac{(1-\omega_2-\omega_2 c_{2m}\beta_2)h_{2m}}{(1-\omega_2)(1-\mu_k c_{2m})} - \frac{(1-\omega_1-\omega_1 c_{1m}\beta_1)}{(1-\omega_1)(1-\mu_k c_{1m})}\left(\frac{T_{\infty 1}}{T_{\infty 2}}\right)^3 h_{2m}\right\} = \left\{\left(\frac{T_{\infty 1}^4}{T_{\infty 2}^4}\right)-1\right\}\Big/4,$$

$$(k=\pm 1, \pm 2, \ldots, \pm n) \tag{13}$$

and

$$g_{jm,k} = \left(\frac{T_{\infty j}}{T_{\infty 2}}\right)^3 \frac{(1-\omega_j-\omega_j c_{jm}\beta_j)}{(1-\omega_j)(1-\mu_k c_{jm})} h_{jm}. \tag{14}$$

The disturbed average radiation intensity and radiation heat flux can be obtained from equations (2) and (3) with the aid of equation (10).

4. PROPAGATION OF PERIODIC DISTURBANCES

For the problem of propagation of small disturbances in a non-scattering radiating gas, the governing equations in discrete ordinate form are

$$\frac{\partial W_S}{\partial t} = -\frac{2\pi\alpha_\infty(\gamma-1)}{\rho_\infty}\left\{\sum_{p=\pm 1}^{\pm n} a_p \frac{\partial I'_P}{\partial t} + \frac{8\sigma T_\infty^3}{\pi R} W_T\right\}, \tag{15}$$

and

$$\frac{\partial}{\partial t}\left\{\mu_k \frac{\partial}{\partial x} + \alpha_\infty\right\} I'_k = -\frac{4\sigma\alpha_\infty T_\infty^3}{\pi R} W_T,$$

$$(k = \pm 1, \pm 2, \ldots, \pm n) \tag{16}$$

which can be combined to give (see Appendix for an alternative derivation)

$$\frac{\partial}{\partial t}\prod_{l=1}^{n} D_l W_S - \frac{16\gamma\alpha_\infty a_S}{Bo}\sum_{l=1}^{n} a_l \mu_l^2 \frac{\partial^2}{\partial x^2}\Big(\prod_{\substack{k=1\\k\neq l}}^{n} D_k W_T\Big) = 0. \tag{17}$$

The boundary conditions for periodic disturbances are

$$\frac{\partial \phi}{\partial x}(0,t) = \frac{RT_\infty}{a_s} Re A \exp(i\omega t), \tag{18}$$

and

$$\frac{\partial I'_k}{\partial t}(0,t) = \frac{4\varepsilon\sigma T_\infty^4 \omega}{\pi} Re\,[B\,e^{i\omega t}] + r_s \frac{\partial I'_k}{\partial t}(0,t)$$

$$- 2r_d \sum_{p=1}^{n} \mu_{-p} a_{-p} \frac{\partial I'_{-p}}{\partial t}(0,t). \tag{19}$$

Equation (19) is obtained by linearizing equation (5) and making use of the boundary condition

$$\frac{dT'_w}{dt} = \omega T_\infty Re\,[B\,e^{i\omega t}]. \tag{20}$$

The constants A and B in equations (18)–(20) are dimensionless complex constants assumed to be given. At infinity, it is required that ϕ is bounded and $I'_k(x)$ must vanish.

The solution of equations (15) and (16) with boundary conditions (18) and (19) can be shown to be

$$\phi = \frac{RT_\infty}{\omega} Re \sum_{m=1}^{n+1} L_m \exp(C_m\xi + i\omega t), \tag{21}$$

and

$$\frac{\partial I'_k}{\partial t} = \frac{4\sigma T_\infty^4 \omega}{\pi} Re \sum_{m=1}^{n+1} F_{mk} \exp(C_m \xi + i\omega t),$$

$$(k = \pm 1, \pm 2, \ldots, \pm n) \tag{22}$$

where C_m are the roots with a positive real part determined from

$$\sum_{p=1}^{n} \frac{a_p}{1 - C_m^2\mu_p^2/Bu^2} = 1 + \frac{iBo(1+C_m^2)}{16\gamma Bu(1+C_m^2/\gamma)}, \tag{23}$$

and L_m and F_{mk} are given by

$$\sum_{m=1}^{n+1} L_m C_m = A, \tag{24}$$

$$\sum_{m=1}^{n+1}\left(1+\frac{C_m^2}{\gamma}\right)\left\{\frac{1}{1+C_m\mu_k/Bu} - \frac{r_s}{1-C_m\mu_k/Bu} - 2r_d\sum_{p=1}^{n}\frac{a_p\mu_p}{1-C_m\mu_p/Bu}\right\} L_m = \varepsilon B,$$

$$(k = 1, 2, \ldots, n) \tag{25}$$

and

$$F_{mk} = \frac{(1 + C_m^2/\gamma) L_m}{(1 + C_m\mu_k/Bu)}. \tag{26}$$

The disturbed pressure, temperature, and density can then be found by a suitable differentiation of equation (21) whereas the disturbed average radiation intensity and radiation heat flux can be obtained from equations (2) and (3) with the aid of equation (22).

5. NUMERICAL RESULTS

Equations (10)–(14) and (21)–(26) were evaluated with Gaussian quadrature up to the third approximation ($n = 3$). For the problem of linearized shock wave, the disturbances in the flow field were computed for the following two cases (with $Bo_1 = 10$ and $\gamma = 1{\cdot}4$): (i) $M_{\infty 1} = 1{\cdot}1$,

Table 1. Disturbed quantities evaluated in front of and behind the shock located at $\eta = 0$

Case		η	$-4·0$	0^-	0^+	$4·0$
	$\frac{T'}{T_{\infty 1}}$	1st approx	0·005157	0·04076	−0·01199	−0·0001940
		2nd approx	0·005273	0·03975	−0·01169	−0·0003266
		3rd approx	0·005269	0·03950	−0·01162	−0·0003256
		exact	0·005268	0·03925	−0·01154	−0·0003263
1	$\frac{I'_0}{4\sigma T^4_{\infty 1}}$	1st approx	0·01875	0·1481	−0·07428	−0·001201
		2nd approx	0·01906	0·1471	−0·07536	−0·001894
		3rd approx	0·01905	0·1469	−0·07551	−0·001891
		exact	0·01905	0·1470	−0·07588	−0·001894
	$\frac{q'}{\sigma T^4_{\infty 1}}$	1st approx	−0·01292	−0·1021	−0·1022	−0·01652
		2nd approx	−0·01321	−0·09961	−0·09962	−0·002781
		3rd approx	−0·01320	−0·09898	−0·09899	−0·002773
		exact	−0·01319	−0·09833	−0·09833	−0·002779
	$\frac{T'}{T_{\infty 1}}$	1st approx	0·01931	0·1041	0·02962	0·0000005332
		2nd approx	0·01899	0·1002	0·02850	0·00004181
		3rd approx	0·01893	0·09971	0·02834	0·00003967
		exact	0·01891	0·09941	0·02827	0·00004071
	$\frac{I'_0}{4\sigma T^4_{\infty 1}}$	1st approx	0·09581	0·5167	−0·07971	−0·000001434
		2nd approx	0·09255	0·5213	−0·07518	0·00002221
		3rd approx	0·09231	0·5226	−0·07381	0·00003098
		exact	0·09213	0·5243	−0·07248	0·00003173
	$\frac{q'}{\sigma T^4_{\infty 1}}$	1st approx	−0·05383	−0·2903	−0·2902	−0·000005224
		2nd approx	−0·05292	−0·2794	−0·2793	−0·0004097
		3rd approx	−0·05276	−0·2778	−0·2777	−0·0003888
		exact	−0·05266	−0·2768	−0·2768	−0·0003968

Table 2. Amplitude of disturbances at the gas–solid interface

		$A = 1, B = 0$		$A = 0, B = 1$	
	Bu	1·0	0·01	1·0	0·01
$\frac{P'}{P_\infty}$	1st approx	0·8827	0·9992	0·5235	0·03458
	2nd approx	0·8880	0·9991	0·4900	0·03456
	3rd approx	0·8885	0·9991	0·4868	0·03455
	exact	0·8887	0·9989	0·4861	0·03402
$\frac{\rho'}{\rho_\infty}$	1st approx	0·8382	0·7147	0·2881	0·0004276
	2nd approx	0·8410	0·7147	0·3038	0·0005729
	3rd approx	0·8416	0·7147	0·3023	0·0006633
	exact	0·8420	0·7147	0·3003	0·001237
$\frac{T'}{T_\infty}$	1st approx	0·06065	0·2850	0·7403	0·03459
	2nd approx	0·06502	0·2849	0·7397	0·03459
	3rd approx	0·06552	0·2849	0·7400	0·03458
	exact	0·06569	0·2847	0·7402	0·03441
$\frac{q'}{4\sigma T^4_\infty}$	1st approx	0·08627	0·005699	0·6683	1·1541
	2nd approx	0·08076	0·005696	0·6353	1·0421
	3rd approx	0·08023	0·005694	0·6311	1·0195
	exact	0·08012	0·005611	0·6283	0·9997
$\frac{I'_0}{16\sigma T^4_\infty}$	1st approx	0·03735	0·002467	0·7591	0·5003
	2nd approx	0·03937	0·003305	0·7577	0·5003
	3rd approx	0·03920	0·003327	0·7576	0·5003
	exact	0·03892	0·007096	0·7575	0·5004

$M_{\infty 2} = 0{\cdot}9118$, $\omega_1 = 0$, $\omega_2 = 0$, (ii) $M_{\infty 1} = 1{\cdot}4$, $M_{\infty 2} = 0{\cdot}7397$, $\omega_1 = 0$, $\omega_2 = 0{\cdot}9$ whereas for the problem of wave propagation, the amplitude of disturbances at the gas–solid interface (for a black surface with $e = 1$) were computed for $Bu = 1{\cdot}0$ and $0{\cdot}01$ with $Bo = 3{\cdot}23$ and $\gamma = 1{\cdot}4$. The results of these calculations along with the exact values obtained by Cheng and Leonard [2, 11] are tabulated in Tables 1 and 2. It is shown in these tables that the results based on the first approximation agree with those of the differential approximation with Mark's boundary condition [2, 11], and that results of higher approximations approach that of the exact solution.

6. CONCLUDING REMARKS

Numerical results accurate to any desired degree of accuracy can be achieved by taking more discrete directions in the radiation field. In view of the availability of electric computers and the convenient form of the method for numerical work, the method of discrete ordinates appears very attractive when a high degree of accuracy is required. The extension of the method to a multi-dimensional radiating flow is now in progress.

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APPENDIX

Alternative derivation of acoustic equation (17)

Further insight can be obtained by an alternative derivation of equation (17). The exact acoustic equation with disturbances generated by a reflecting wall at $x = 0$ is given by (12)

$$\left\{\frac{\partial W_S}{\partial t} + \frac{16\gamma\alpha_\infty a_S}{Bo} W_T\right\} = \frac{8\gamma a_S \alpha_\infty}{Bo} \Bigg[\alpha_\infty \int_0^x W_T(x')\, E_1\{\alpha_\infty(x - x')\}\,\mathrm{d}x' + \alpha_\infty \int_x^\infty W_T(x')\, E_1\{\alpha_\infty(x' - x)\}\,\mathrm{d}x' - \left\{\varepsilon R\,\mathrm{d}T_w/\mathrm{d}t - 2r_d\alpha_\infty \int_0^\infty W_T(x')\, E_2(\alpha_\infty x')\,\mathrm{d}x'\right\} E_2(\alpha_\infty x) + r_s\alpha_\infty \int_0^\infty W_T(x')\, E_1\{\alpha_\infty(x' + x)\}\,\mathrm{d}x'\Bigg]. \tag{A.1}$$

If we apply equation (1) to the integro-exponential function, $E_m(z)$, namely, letting

$$E_m(z) \equiv \int_0^1 \mu^{m-2} \exp(-z/\mu)\,\mathrm{d}\mu \simeq \sum_{p=1}^{n} a_p(\mu_p)^{m-2} \exp(-z/\mu_p), \tag{A.2}$$

and substitute equation (A.2) into equation (A.1), the resulting equation after $2n$ differentiation with respect to x leads to

$$\prod_{l=1}^{n} D_l \left\{\frac{\partial W_s}{\partial t} + \frac{16\gamma a_S \alpha_\infty W_T}{Bo}\right\} = \frac{16\gamma a_s \alpha_\infty^3}{Bo} \sum_{l=1}^{n} a_l \prod_{\substack{k=1\\k\neq l}}^{n} D_k W_T, \tag{A.3}$$

which can easily be rearranged to give equation (17). It should be noted that the approximate formula similar to equation (A.2) with the constants a_p and μ_p determined rather arbitrarily were proposed by various authors [14, 15]. For the first approximation equation (A.2) gives $E_m(z) = (1/\sqrt{3})^{m-2} \exp[-(\sqrt{3})z]$ and equation (A.3) reduces to the same equation obtained by Cheng [16]. The discrete ordinate method thus gives the approximation, equation (A.2), a new mathematical interpretation and offers the determination of the constants on a rigorous basis.